Continuous time and Discrete time Signals and Systems

1. Systems in Engineering

A system is usually understood to be an engineering device in the field, and a mathematical representation of this system is usually called a system model. It can be defined as the mathematical relationship between an input signal and an output signal. The word system refers to many different things in engineering. It may be refers to:

- Tangible objects: such as software systems, electronic systems, computer systems, or mechanical systems.
- Theoretical objects: such as a system of linear equations or a mathematical input-output model.

2. Signals

Signals are functions of time that represent the evolution of variables and describe a wide variety of physical phenomena. Signals carry information about these physical phenomena in a pattern of variation of some forms. Signals are represented mathematically as functions of one (one dimensional signal) or more independent variables (multi-dimensional signal). It can be classified from the independent variable nature into two types:

- **Continuous Time (CT):** If the independent variable is continuous this refers to Continuous Time signal $x(t)$. These signals are defined for continuum of values of the independent variable.

- **Discrete Time (DT):** signals $x[n]$ are defined only at discrete times and the independent variable takes only a discrete set of values. They are defined only for integer values of the independent variable (time steps).

Consider the simple circuit shown in Fig.1 the pattern of variation over time in the source and capacitor voltage $V_s$ and $V_C$. Similarly, in Fig. 2, the variation of the applied force $f$ over time resulting in automobile velocity $V$. 

Fig. 1: A simple RC circuit

Fig. 2: An automobile responding to a force f

Fig. 3: Recording of human speech

Fig. 4: A monochromatic picture

Fig. 5: Annual vertical wind profile

Fig. 6: A discrete time signal

Fig. 7: A Continuous Time signal and discrete Time signal

- Fig. 3 is an illustration of a recording of human speech signal using microphone to sense acoustic pressure. *Speech → acoustic pressure (time)*

- The *monochromatic picture* shown in Fig. 4, it represents the pattern of variation in brightness versus the (x, y) position of the image. *Brightness → f(spatial variables x, y).*
• In geophysics, signals representing variations with depth of physical quantities such as density, porosity, and electrical resistivity are used to study the structure of the earth.
• In metrological investigations, the knowledge of air pressure, temperature and wind speed with latitude are extremely important. Fig. 5 describes an annual average of vertical wind profile as function of height. Used in weather patterns and aircraft landing and final approach. Fig. 7 represents a graphical illustration of CT and DT signal.

*How to obtain discrete signal from continuous signal?*

Discrete signals often arise from signals with continuous domains by sampling. Continuous domains have an infinite number of elements. Even the domain $x \in [0,1]$ Time representing a finite time interval has an infinite number of elements. The signal assigns a value in its range to each of these infinitely many elements. Such a signal cannot be stored in a finite digital memory device such as a computer or CD-ROM. If we wish to store, say, Voice, we must approximate it by a signal with a finite domain. A common way to approximate a function with a continuous domain like Voice and Image by a function with a finite domain is by *uniformly sampling* its continuous domain.

![Fig. 8: The exponential functions $e^x$ and Sampled Exp, with sampling interval of 0.2.](image)

If we sample a 10-second long domain of Voice $[0,10]$, 10,000 times per second (i.e. at a frequency of 10 kHz) we get the signal \( \text{Sampled Voice} = \{0,0.0001,0.0002, \ldots , 9.9998, 9.9999,10\} \). In the example, the sampling interval or *sampling period* is 0.0001 sec, corresponding to a *sampling frequency* or sampling rate of 10,000 Hz. Since the continuous domain is 10 seconds long, the domain of Sampled Voice has 100,000 points. A sampling frequency of 5,000 Hz would give the Sampled Voice $= \{0,0.0002,0.0004, \ldots , 9.9998,10\}$,
which has half as many points. The sampled domain is finite, and its elements are discrete values of time.

Fig. 8 shows an exponential function $e^x$ and its sampled signal with sampling interval = 0.2

Continuous-time and discrete-time functions map their domain (time interval) into their co-domain (range or set of values). This is expressed in mathematical notation as $f: T \rightarrow V$. As shown in Fig. 9.

![Fig. 9: Domain, co-domain, and range of a real function of continuous time.](image)

3. Transformation of the Independent Variable

Consider the continuous-time signal $x(t)$ defined by its graph shown in Fig. 10 and the discrete-time signal $x[n]$ defined by its graph in Fig. 11. As an aside, these two signals are said to be of finite support, as they are nonzero only over a finite time interval, namely on $t \in [-2,2]$ for $x(t)$ and $n \in \{-3,\ldots,3\}$ when for $x[n]$. We will use these two signals to illustrate some useful transformations of the time variable, such as time scaling and time reversal.

- **Time Scaling**

  Time scaling refers to the multiplication of the time variable by a real positive constant $\alpha$.

  **In the continuous-time case**, we can write $y(t) = x(\alpha t)$

  **In case $0 < \alpha < 1$**: The signal $x(t)$ is slowed down or expanded (stretched) in time. Think of a tape recording played back at a slower speed than the nominal speed.

  **In case $\alpha > 1$**: The signal $x(t)$ is sped up or compressed in time. Think of a tape recording played back at twice the nominal speed.

  Try to guess the behavior in case $\alpha < 0$ ..........................................................
In the discrete time case, we can write $y[n] = x[an]$

In case $\alpha > 1$: where $\alpha$ is an integer, makes sense, as $x[n]$ is undefined for fractional values of $n$. In this case, called *decimation or down-sampling*, we not only get a time compression of the signal, but the signal can also lose part of its information; that is, some of its values may disappear in the resulting signal $y[n]$. 

In case $\alpha < 1$: where $\alpha$ is a non-integer, makes sense, as $x[n]$ is undefined for fractional values of $n$. In this case, called *interpolation or up-sampling*, we not only get a time expansion of the signal, but the signal can also gain part of its information; that is, some of its values may appear in the resulting signal $y[n]$. 

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**Time Reversal**

A time reversal is achieved by multiplying the time variable by $-1$. If $x(t)$ represents an audio tap recording, then $x(-t)$ is the same tap recording played back word.
A continuous Time Signal $x(t)$

A Discrete Time Signal $x[n]$  

$y(t) = x(-t)$  

$y[n] = x[-n]$  

Fig. 11: Transformation of the independent variable (Time Reversal) on CT and DT signal

- **Time Shift**

A time shift *delays or advances* the signal in time by a continuous-time interval $y(t) = x(t + T)$ For $T$ positive, the signal is advanced; For $T$ negative, the signal is delayed. Similarly, a time shift delays or advances a discrete-time signal by an integer discrete-time interval $N$: $y[n] = x[n + N]$ For $N$ positive, the signal is advanced by $N$ time steps, For $N$ negative, the signal is delayed by $|N|$ time steps.

Several receivers at different locations observe a signal being transmitted through a medium (water, rock, air,..., etc.). In this case, the difference in propagation time from the origin of the transmitted signal to any receivers results in a time shift between the signals at the two receivers.
4. Periodic Signals

A periodic continuous time signal is a signal that has the following property:

\[ x(t) = x(t + T) \text{ where } T \text{ is a positive value.} \]

\[ x(t) = x(t + mT) \text{ where } T \text{ is appositive value, } m \text{ is integer value.} \]

Or it is unchanged by a time shift of T. The fundamental period \( T_0 \) is the smallest positive value that achieves the periodicity of a signal.

If \( x(t) \) is constant, can you define the fundamental period?

If \( x(t) \) is not periodic then \( x(t) \) is **non-periodic** signal.

A periodic discrete time signal is a signal that has the following property:

\[ x[n] = x[n + N] \]
The fundamental period $N_0$ is the smallest positive value of $N$ that achieves the periodicity of a signal. Examples of periodic continuous and discrete time signals are shown in Fig. 11.

5. Even and Odd signals

A signal $x(t)$ or $x[n]$ is referred to as an even signal if it is identical to its time reversal

$$x(-t) = x(t) \text{ or } x[-n] = x[n]$$

A signal is referred to as odd if

$$x(-t) = -x(t) \text{ or } x[-n] = -x[n]$$

- An important fact that any signal $x(t)$ can be broken to the sum of two signals one of them is even signal $x_e(t)$ and the other is odd signal $x_o(t)$

$$x(t) = x_e(t) + x_o(t)$$

Where $x_e(t) = \frac{1}{2}(x(t) + x(-t))$ the even part of $x(t)$ and

$$x_o(t) = \frac{1}{2}(x(t) - x(-t))$$ the odd part of $x(t)$
Fig. 15: an example of the even and odd decomposition of discrete time signal
**Exercise 1:** A continuous-time signal $x(t)$ is shown in Fig. Sketch and label each of the following signals:

- $x(t - 2)$
- $x(2t)$
- $x(t/2)$
- $x(-t)$

$x(t - 2)$ is the same as $x(t)$ with delay of two units

$x(2t)$: $x(t)=0$ for all $t \leq 0$, at $t=1$ $x(2t)=x(2)$, at $t=2$ $x(2t)=x(4)$, at $t \geq 3$ $x(2t) = x(6) = x(8) = \ldots = 0$

$x(t/2)$: $x(t)=0$ for all $t \leq 0$, at $t=1$ $x(t/2)=x(1/2)$, at $t=2$ $x(t/2)=x(1)$, at $t=3$ $x(t/2)=x(3/2)$, at $t=4$ $x(t/2)=x(2)$, \ldots

At $x(-t)$
Exercise 2: A discrete-time signal $x[n]$ is shown in Fig. Sketch and label each of the following signals.

\[ x[n - 2] , \]
\[ x[2n] , \]
\[ x[-n] , \]
\[ x[-n + 2] \]

At $n=0$  
\[ x[0-2]=x[-2]=0 \]
At $n=1$  
\[ x[-1]=0, \quad x[0]=0, \quad x=3=x[1]=1 \]
\[ x=4=x[2]=2 \] ....

At $x=0$  
\[ x[0]=0 \]
At $x=1$  
\[ x[2*1]=x[2]=2 \]
At $x=2$  
\[ x[4]=3 \]
At $x=3$  
\[ x[6]=0 \]
Exercise 3: Given the continuous-time signal specified by \( x(t) = \begin{cases} 1 - |t| & -1 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} \) sketch this signal and determine the resultant discrete-time sequence obtained by uniform sampling of \( x(t) \) with a sampling interval of (a) 0.25 s, (b) 0.5 s, and (c) 1.0 s.

Solution: It is easier to take the graphical approach for this problem. The signal \( x(t) \) is plotted in figure (a). Figures (b), (c), (d) give plots of the resultant sampled sequences obtained for the three specified sampling intervals 0.25 s, 0.5 s, and 1.0 s.

At
\[ T_s = 0.25 \text{ s}: \ x[n] = \{0,0,0.25,0.5,0.75,1,0.75,0.5,0.25,0,0\} \]

At \( T_s = 0.5 \text{ s}: \ x[n] = \{\ldots,0,0,0.5,1,0.5,0,0\ldots\} \)

At \( T_s = 1 \text{ s}: \ x[n] = \{\ldots,0,0,1,0,0\ldots\} \)

Exercise 4: if \( x_1[n] = \{0,0,1,2,3,0,0,2,2,0\} \) and \( x_2[n] = \{0,-2,-2,2,2,0,0,-2,0\} \)

Sketch \( x_1[n] \), \( x_2[n] \) and represent each of the following signals by a graph and by a sequence of numbers: \( y_1[n] = x_1[n] + x_2[n] \), \( y_2[n] = 2x_1[n] \), \( y_3[n] = x_1[n] x_2[n] \)

Solution:

Graphic representation of \( x_1[n], x_2[n] \)

\( y_1[n] = x_1[n] + x_2[n] = \{\ldots,0,-2,-2,3,4,3,-2,0,2,2,0\ldots\} \)

\( y_2[n] = 2x_1[n] = \{0,0,2,2,4,6,0,0,4,4,0\} \)
\[
y_3[n] = x_1[n] x_2[n] = \{0,0,2,4,0,0\}
\]

Graphic representation of \(y_1[n], y_2[n]\) and \(y_3[n]\)

**Exercise 5:** Find the even and odd components of \(x(t) = e^{jt}\)

**Solution:**

\[
x(t) = x_e(t) + x_o(t)
\]

Where \(x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}, x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}\)

\[
x_e(t) = \frac{1}{2}\{x(t) + x(-t)\} = \frac{1}{2}\{e^{jt} + e^{-jt}\} \quad \text{Using Euler's formula, we obtain}
\]

\[
= \frac{1}{2}\{\cos t + j \sin t + \cos t - j \sin t\} = \frac{1}{2}\{2 \cos t\} = \cos t
\]

\[
x_o(t) = \frac{1}{2}\{x(t) - x(-t)\} = \frac{1}{2}\{e^{jt} - e^{-jt}\}
\]

\[
= \frac{1}{2}\{\cos t + j \sin t - \cos t - j \sin t\} = \frac{1}{2}\{2 j \sin t\} = j \sin t
\]

By substituting \(x_e(t), x_o(t)\) in Eq. 11

\[
x(t) = e^{jt} = \cos t + j \sin t
\]
**Exercise 6:** Sketch and label the even and odd components of the following Signals in Figure.

**Solution:**

Exercise 6 solution: Odd and even components of signals in Exercise 6
**Exercise 7:** Show that the complex exponential signal \( x(t) = e^{j\omega_0 t} \) is periodic and that its fundamental period is \( T_0 = \frac{2\pi}{\omega_0} \).

**Solution:** \( x(t) = e^{j\omega_0 t} \) is a periodic signal if
\[
x(t) = x(t + T) \Rightarrow e^{j\omega_0 t} = e^{j\omega_0 (t + T)} = e^{j\omega_0 t} e^{j\omega_0 T}
\]
This can be achieved if \( e^{j\omega_0 T} = 1 \Rightarrow \omega_0 T = 0 \)

If \( \omega_0 = 0 \) then \( e^{j\omega_0 T} = 1 \) and \( x(t) = x(t + T) \) is periodic.

If \( \omega_0 \neq 0 \) then \( e^{j\omega_0 T} = 1 \) only if \( \omega_0 T = m2\pi \) or \( m = \frac{2\pi}{\omega_0} \), if \( m = 1 \) thus the fundamental period of \( x(t) \) is \( T_0 = \frac{2\pi}{\omega_0} \).

**Exercise 8:** Show that the complex sinusoidal signal \( x(t) = \cos(\omega_0 t + \theta) \) is periodic and that its fundamental period is \( \frac{2\pi}{\omega_0} \).

**Solution:** The sinusoidal signal \( x(t) = \cos(\omega_0 t + \theta) \) is a periodic signal if:
\[
\cos[\omega_0 (t + T) + \theta] = \cos(\omega_0 t + \theta)
\]
\[
\cos[\omega_0 (t + T) + \theta] = \cos(\omega_0 t + \theta + \omega_0 T) = \cos(\omega_0 t + \theta)
\]
which can be achieved only if
\[
\omega_0 T = 0 \text{ or } \omega_0 T = m \cdot 2\pi \text{ as in Ex. 5.}
\]

**Exercise 9:** Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

\[
\begin{align*}
(a) \quad x(t) &= \cos(t + \frac{\pi}{4}) \\
(b) \quad x(t) &= \sin \frac{2\pi}{3} t \\
(c) \quad x(t) &= \cos \frac{\pi}{3} t + \sin \frac{\pi}{4} t \\
(d) \quad x(t) &= \cos t + \sin \sqrt{2} t \\
(e) \quad x(t) &= \sin^2 t \\
(f) \quad x(t) &= e^{j(\pi/2)t - 1}
\end{align*}
\]

**Solution:**

(a) \( x(t) = \cos(t + \frac{\pi}{4}) \) is a sinusoidal signal in the form \( x(t) = \cos(\omega_0 t + \theta) \) where \( \omega_0 = 1 \) and \( \theta = 0 \)

The fundamental period \( T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{1} = 2\pi \)

\( x(t) \) is periodic and The fundamental period \( T_0 = 2\pi \).
(b) Try to solve it as in the section (a)

(c) \( x(t) = \cos \frac{\pi}{3} t + \sin \frac{\pi}{4} t = x_1(t) + x_2(t) \)

Check the periodicity of \( x_1(t) \) and find the fundamental period \( T_{0-1} \)

Check the periodicity of \( x_2(t) \) and find the fundamental period \( T_{0-2} \)

Check if \( \frac{T_{0-1}}{T_{0-2}} \) is a rational number then \( x(t) \) is periodic signal.

(d) \( x(t) = \cos t + \sin \sqrt{2} t \quad \text{Try to solve it as in the section (c)} \)

(e) \( x(t) = \sin^2 t = \frac{1 - \cos 2t}{2} = \frac{1}{2} - \frac{\cos 2t}{2} = x_1(t) + x_2(t) \quad \text{complete the solution as in section (c) and (d).} \)

(f) \( x(t) = e^{j(\pi/2)t-1} = e^{j(\pi/2)t}e^{-j} = e^{j\omega_0 t}e^{-j} \quad \text{where} \omega_0 = \frac{\pi}{2} \)

\( e^{j\omega_0 t} \) is aperiodic signal with fundamental period \( T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi/2} = 4 \), \( e^{-j} \) is constant and does not affect the periodicity of \( x(t) \).

**Exercise 10:** Determine whether the following are energy signals, power signals, or neither

(a) \( x(t) = e^{-at}u(t), \quad a > 0 \)

(b) \( x(t) = A \cos(\omega_0 t + \theta) \)

(c) \( x(t) = tu(t) \)

**Solution:**

(a) \( E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \)

\[ \int_{0}^{+\infty} e^{-2at} dt = \left. \frac{1}{-2a} [e^{-2at}] \right|_{0}^{\infty} = \frac{1}{-2a} [e^{-2a\infty} - e^{-2a0}] = \frac{1}{2a} [e^{-\infty} - e^{0}] = \frac{1}{2a} < \infty \]

Thus \( x(t) \) is an energy signal.

(b) The sinusoidal signal \( x(t) \) is periodic with period with \( T_0 = \frac{2\pi}{\omega_0} \), the average power of \( x(t) \) is

\[ P = \frac{1}{T_0} \int_{0}^{T_0} |x(t)|^2 dt = \frac{w_0}{2\pi} \int_{0}^{\frac{2\pi}{\omega_0}} A^2 \cos^2(w_0 t + \theta) dt \]

\[ = A^2 \frac{w_0}{2\pi} \int_{0}^{\frac{2\pi}{\omega_0}} \frac{1 + \cos(2w_0 t + 2\theta)}{2} dt \]

\[ = A^2 \frac{w_0}{4\pi} \int_{0}^{\frac{2\pi}{\omega_0}} 1 + \cos(2w_0 t + 2\theta) dt \]
Signals and Systems  
Solved Exercise on Lecture 1

\[ P = \frac{A^2w_0}{4\pi} \left( \int_{0}^{\frac{2\pi}{w_0}} 1 \, dt + \int_{0}^{\frac{2\pi}{w_0}} \cos(2w_0t + 2\theta) \, dt \right) = \frac{A^2w_0}{4\pi} [I_1 + I_2] \quad \text{eqn (1)} \]

\[ I_1 = \int_{0}^{\frac{2\pi}{w_0}} 1 \, dt = \left| t \right|_{0}^{\frac{2\pi}{w_0}} = \frac{2\pi}{w_0} - 0 = \frac{2\pi}{w_0} \quad \text{eqn (2)} \]

\[ I_2 = \int_{0}^{\frac{2\pi}{w_0}} \cos(2w_0t + 2\theta) \, dt = \frac{1}{2w_0} \sin(2w_0t + 2\theta) \bigg|_{t=0}^{t=\frac{2\pi}{w_0}} \]

\[ I_2 = \frac{1}{2w_0} \left( \sin\left(2w_0 + 2\theta\right) - \sin(2\theta)\right) \]

\[ I_2 = \frac{1}{2w_0} (\sin(4\pi + 2\theta) - \sin(2\theta)) = 0 \quad \text{eqn (3)} \]

Substitute using eqn 2 and eqn 3 in eqn 1

\[ P = \frac{A^2w_0}{4\pi} [I_1 + I_2] = \frac{A^2w_0}{4\pi} \left[ \frac{2\pi}{w_0} + 0 \right] = \frac{A^2}{2} \]

Thus \( x(t) \) is power signal, and in general periodic signals are power signal.

\[(c) \quad E = \lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|^2 \, dt = \lim_{T \to \infty} \int_{0}^{T/2} t^2 \, dt = \lim_{T \to \infty} \frac{(T/2)^3}{3} = \infty \]

\[ P = \lim_{T \to \infty \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt = \lim_{T \to \infty \frac{1}{T} \int_{0}^{T/2} t^2 \, dt = \lim_{T \to \infty \frac{1}{T} \frac{(T/2)^3}{3} = \infty} \]

Thus, \( x(t) \) is neither an energy signal nor a power signal.